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CHEBYSHEV RANK IN L_1-APPROXIMATION

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ABSTRACT. Let $C_\omega(K)$ denote the space of continuous functions endowed with the norm $\int_K \omega|f| = \|f\|_\omega$, $\omega > 0$. In this paper we characterize the subspaces $U_n \subset C_\omega(K)$ having Chebyshev rank at most k $(0 \le k \le n-1)$ with respect to all bounded positive weights ω. Various applications of main results are also presented.

0. Introduction. Let X be a real normed linear space and consider a finite dimensional subspace $U_n \subset X$, $\dim U_n = n$. For each $f \in X$ denote by $Y(f)$ the set of best approximants of f in U_n, that is $Y(f)$ is the union of all those $p \in U_n$ for which $\|f-p\| = \inf\{\|f-q\| : q \in U_n\}$. Evidently, for any $f \in X$ the set $Y(f)$ is nonempty and convex. As usual, we say that $Y(f)$ has dimension k $(0 \le k \le n)$ if there exist $k+1$ elements $q_0, \dots, q_k$ in $Y(f)$ such that $q_1 - q_0, \dots, q_k - q_0$ are linearly independent and we cannot find $k+2$ elements in U_n with this property. Furthermore, the maximal dimension of $Y(f)$ over all $f \in X$ is called the Chebyshev rank of U_n in X.

The case of Chebyshev rank being equal to 0 corresponds to the situation when each element of X has unique best approximant in U_n. In this case U_n is called a Chebyshev subspace of X.

A classical result of Haar [**3**] gives a complete characterization of Chebyshev subspaces in $C(K)$, the space of real continuous functions endowed with supremum norm on the compact Hausdorff set K. According to the Haar's theorem U_n is a Chebyshev subspace of $C(K)$ if and only if each $q \in U_n \backslash \{0\}$ has at most $n-1$ zeros at K. In what follows we shall say that an n-dimensional subspace U_n in $C(K)$ is a Haar space on K if its elements have at most $n-1$ zeros on K. Rubinstein [**16**] gave an elegant generalization of Haar's theorem showing that $U_n \subset C(K)$ has Chebyshev rank at most $k-1$ in $C(K)$, $1 \le k \le n$, if and only if any k linearly independent elements of U_n have at most $n-k$ common zeros on K. Analogous results for the space of differentiable functions were obtained in [**2** and **11**].

In the present paper we shall consider a similar problem with respect to the L_1-norm. It can be easily seen that in $X = L_1[a,b]$ any subspace U_n has Chebyshev rank n. Indeed, according to the Hobby-Rice theorem [**6**] there exist points $a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b$ $(0 \le k \le n)$ such that

$$(1) \qquad \sum_{j=0}^{k} (-1)^j \int_{x_j}^{x_{j+1}} q = 0, \qquad q \in U_n.$$

Then setting $f(x) = \operatorname{sign} \prod_{j=1}^{k} (x - x_j) \in L_1[a,b]$ it can be easily shown that $Y(f)$ contains the whole unit ball of U_n, i.e., $\dim Y(f) = n$. This result is essentially

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due to Krein [**9**]. Thus it is more natural to study L_1-approximation of continuous functions. Let $C_\omega(K)$ be the space of real continuous functions on the compact set $K \subset \mathbf{R}^s$ ($s \in \mathbf{N}$) endowed with the weighted L_1-norm $\|f\|_\omega = \int_K \omega|f|$. Here $\omega \in W$, where W denotes the set of all measurable weights ω such that $0 < \inf_{x\in K}\omega(x) \leq \sup_{x\in K}\omega(x) < \infty$. In what follows we shall always assumee that $K = \overline{\operatorname{Int} K}$. By a famous result of Jackson [**7**] and Krein [**9**], if $U_n \subset C[a,b]$ is a Haar space on (a,b) then it is a Chebyshev subspace of $C_\omega[a,b]$, $\omega \in W$. However, it turned out that the Haar property is not necessary in general for uniqueness of best L_1-approximation of continuous functions. Different criteria for Chebyshev subspaces in $C_\omega(K)$ for a given $\omega \in W$ were found by a number of authors, see, e.g., [**1**, **20**], but unlike the situations in uniform approximation these criteria depend on the weight ω and therefore are not at all easily checked for specific subspaces.

Recently it was proved that the so-called A-spaces give a complete weight independent characterization of Chebyshev subspaces in $C_\omega(K)$. Let us give the corresponding definition. We shall say that ψ is a sign function supported by an open subset T of K, written $\operatorname{supp}\psi = T$, if $|\psi| = \xi_T$ (the characteristic function of T) and ψ is continuous on T. $Z(f) = \{x \in K : f(x) = 0\}$ will denote the set of zeros of the function f.

DEFINITION 1. The subspace $U_n \subset C(K)$ is called an A-space if for any $p \in U_n\backslash\{0\}$ and any sign function ψ with $\operatorname{supp}\psi = K\backslash Z(p)$ we can find $q \in U_n\backslash\{0\}$ such that $\psi q \geq 0$ on K and $q = 0$ a.e. on $Z(p)$.

The notion of A-spaces was introduced with a suggestion of De Vore by Strauss [**19**]. Evidently, if $U_n \subset C[a,b]$ is a Haar space on (a,b) then it satisfies the A-property. It is also known that different families of spline functions satisfy the A-property. Pinkus [**14**] gave a complete characterization of A-spaces when $K = [a,b]$.

The next result can be considered as the analogue of Haar's theorem in L_1.

THEOREM 1 (STRAUSS [**19**], KROÓ [**12**], PINKUS [**14**]). *In order for U_n to be a Chebyshev subspace of $C_\omega(K)$ for each $\omega \in W$ it is necessary and sufficient that U_n is an A-space.*

The sufficiency in the above result was proved by Strauss [**19**]. Necessity was verified independently by Pinkus [**14**] and the author [**12**]. (A special case was verified earlier by Havinson [**4**].) Recently, Theorem 1 was generalized by the author for Banach space valued functions [**13**].

1. Main results. Our main goal is to extend Theorem 1 using the notion of Chebyshev rank; that is, we shall give a weight independent analogue of Rubinstein's theorem in the L_1-norm. (Chebyshev rank in L_1-approximation with respect to a given weight was studied by Havinson [**5**].) Let us denote by $Z(p_1,\dots,p_r) = \bigcap_{i=1}^r Z(p_i)$ the set of common zeros of the functions $p_1,\dots,p_r$.

DEFINITION 2. The subspace $U_n \subset C(K)$ is called an A^k-space $(0 \leq k \leq n-1)$ if for any $k+1$ linearly independent elements $q_0, q_1, \dots, q_k \in U_n$ and any sign function ψ with $\operatorname{supp}\psi = K\backslash Z(q_0,\dots,q_k)$ there exists $q \in U_n\backslash\{0\}$ such that $\psi q \geq 0$ on K and $q = 0$ a.e. on $Z(q_0,\dots,q_k)$.

Thus the notion of A^k-spaces is a natural extension of the A-property. In particular, for $k = 0$ the A^k-property reduces to the A-property.

Our main result is the following

THEOREM 2. *In order for U_n to have Chebyshev rank at most k in $C_\omega(K)$ for each $\omega \in W$ it is necessary and sufficient that U_n is an A^k-space* $(0 \le k \le n-1)$.

Theorem 2 extends Theorem 1 in the spirit of Rubinstein. We shall present various corollaries and applications of Theorem 2. We would like to mention here one interesting consequence of Theorem 2.

THEOREM 3. *Let $K = K_m = \bigcup_{j=1}^m [\alpha_j, \beta_j]$, where $\alpha_1 < \beta_1 < \alpha_2 < \cdots < \alpha_m < \beta_m$, and assume that $U_n \subset C(K_m)$ is a Haar space on $\bigcup_{j=1}^m (\alpha_j, \beta_j)$, $1 \le m \le n$. Then the Chebyshev rank of U_n in $C_\omega(K_m)$ is at most $m-1$ for any $\omega \in W$.*

This result extends the Jackson-Krein theorem to disjoint intervals. The proof of Theorem 3 will be based on Theorem 2 and an important property of Haar spaces at disjoint intervals which is believed to be new.

In §2 we shall prove a lemma from the moment theory which plays a crucial role in verification of the necessity in Theorem 2. §3 is devoted to the proof of Theorem 2. In §4 we discuss properties of Haar spaces on disjoint intervals and present a proof of Theorem 3. We also verify that Theorem 3 gives a sharp estimate for Chebyshev rank. Finally, in §5 we discuss some applications of the main results.

2. A lemma from the moment theory. Let $U_n = \operatorname{span}\{\varphi_1, \ldots, \varphi_n\}$ be an n-dimensional subspace of $L_1(S)$, where $S \subset \mathbf{R}^s$ is a measurable bounded set. The moment theory is devoted to the study of the set of n-tuples

$$A_n = \left\{ \left(\int_S \omega \varphi_i \right)_{i=1}^n : \omega \in \tilde{W} \right\} \subset \mathbf{R}^n, \tag{2}$$

where $\tilde{W}$ is a given subset of W. We shall verify that if U_n does not contain a nontrivial element which is nonnegative a.e. on S and $\tilde{W}$ satisfies certain properties, then $A_n = \mathbf{R}^n$. A special case of this statement can be found implicitly in Krein-Nudelman [**10**, pp. 27–28] (see also [**15**] where it was rediscovered for $U_n \subset C[a,b]$). Below we shall present it in a more general form.

The properties of $\tilde{W}$ which are required are as follows:

(i) $\tilde{W}$ is a convex cone, i.e., $\alpha\omega_1 + \beta\omega_2 \in \tilde{W}$ if $\omega_1, \omega_2 \in \tilde{W}$ and $\alpha, \beta > 0$;

(ii) if $\omega_1, \omega_2 \in \tilde{W}$, then $\omega_1 - \varepsilon\omega_2 \in \tilde{W}$ if $\varepsilon > 0$ is small enough;

(iii) if for a given $q \in U_n$ and every $\omega \in \tilde{W}$ we have $\int_S \omega q \ge 0$, then $q \ge 0$ a.e. on S.

LEMMA 1. *Let $U_n = \operatorname{span}\{\varphi_1, \ldots, \varphi_n\} \subset L_1(S)$ and assume that no $q \in U_n \backslash \{0\}$ is nonnegative a.e. on S. If the subset $\tilde{W}$ of W satisfies* (i)–(iii), *then $A_n = \mathbf{R}^n$.*

PROOF. It follows from (i) that A_n is a convex cone as well. Since U_n does not contain a nonnegative element property, (iii) of $\tilde{W}$ implies that A_n intersects any hyperplane $H_d = \{a \in \mathbf{R}^n : \langle a, d \rangle = 0\}$, $d \in \mathbf{R}^n$, and $\operatorname{Int} A_n \ne \emptyset$. We state that, in addition, A_n is open. Indeed, since $\dim A_n = n$ there exist $\omega_1, \ldots, \omega_n \in \tilde{W}$ such that $l_j = (\int_S \omega_j \varphi_i)_{i=1}^n$, $1 \le j \le n$, are linearly independent. Consider an arbitrary $l \in A_n$, $l = (\int_S \omega \varphi_i)_{i=1}^n$, $\omega \in \tilde{W}$. We have by (i) and (ii) that $\omega \pm \varepsilon\omega_j \in \tilde{W}$ $(1 \le j \le n)$ if $\varepsilon > 0$ is small enough. Therefore $l \pm \varepsilon l_j \in A_n$, $1 \le j \le n$. Thus, linear independence of the l_j's and convexity of A_n imply that A_n contains an n-dimensional ball with center at l, that is A_n is open. Now in order to prove the lemma it suffices to show that $0 \in A_n$. If to the contrary $0 \in \operatorname{Bd} A_n$, then the hyperplane supporting A_n at 0 should intersect A_n at some point, contradicting the fact that A_n is open. The lemma is proved.

Lemma 1 unveils an interesting connection between the Hobby-Rice theorem (see (1)), and WT-spaces. Recall that $U_n \subset C[a,b]$ is said to be a WT-space if each $q \in U_n \backslash \{0\}$ has at most $n-1$ sign changes in (a,b). By the Hobby-Rice theorem for each $\omega \in W$ there exist points $a = x_0 < \cdots < x_k < x_{k+1} = b$ $(0 \leq k \leq n)$ such that

$$\sum_{j=0}^{k} (-1)^j \int_{x_j}^{x_{j+1}} \omega q = 0, \qquad q \in U_n. \tag{3}$$

This raises the following question: For which spaces $U_n \subset C[a,b]$ are the number of points $x_1, \ldots, x_k$ satisfying (3) not less than n for every positive continuous weight? It turns out that this is a characteristic property of WT-spaces. Let $W_C = W \cap C[a,b]$ be the set of positive continuous weights.

PROPOSITION 1. *Let U_n be an n-dimensional subspace of $C[a,b]$. Then the following statements are equivalent:*

(a) *U_n is a WT-space;*

(b) *for any $\omega \in W_C$ the number of points in* (3) *is exactly n.*

PROOF. By a result of Jones and Karlovitz [**8**] U_n is a WT-space if and only if for any $a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b$, where $0 \leq k \leq n-1$, there exists $q \in U_n \backslash \{0\}$ satisfying

$$(-1)^i q(x) \geq 0, \qquad x \in [x_{i-1}, x_i] \ (1 \leq i \leq k+1). \tag{4}$$

This immediately verifies the well-known implication (a) $\Rightarrow$ (b).

Assume now that U_n is not a WT-space. Again using the Jones-Karlovitz theorem we can conclude that for some $a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b$ $(0 \leq k \leq n-1)$ there does not exist $q \in U_n \backslash \{0\}$ for which (4) holds. Define the function ψ by $\psi(x) = (-1)^i$, $x \in [x_{i-1}, x_i)$ $(1 \leq i \leq k+1)$ and set $\tilde{U}_n = \{\psi q : q \in U_n\}$. Obviously, $\tilde{U}_n$ does not contain nonnegative elements (except the zero element), hence by Lemma 1 there exists $\omega \in W_C$ such that $\int_a^b \omega \psi q = 0$ for each $q \in U_n$, i.e., (3) holds for $x_1, \ldots, x_k$, where $k \leq n-1$. This completes the proof of the proposition.

3. Proof of Theorem 2.

Sufficiency. Assume that for some $\omega \in W$ the Chebyshev rank of U_n in $C_\omega(K)$ is larger than k, i.e., we can find $f \in C_\omega(K)$ for which $\dim Y(f) \geq k+1$. Then there exist $q_0, q_1, \ldots, q_{k+1} \in U_n$, all of them best approximants of f, such that $q_i^* = q_i - q_0 \in U_n$ $(1 \leq i \leq k+1)$ are linearly independent. Since $(1/(k+2)) \sum_{i=0}^{k+1} q_i$ is also a best approximant of f, by a standard argument we obtain that a.e. on K

$$\left| f - \frac{1}{k+2} \sum_{i=0}^{k+1} q_i \right| = \frac{1}{k+2} \sum_{i=0}^{k+1} |f - q_i|. \tag{5}$$

The continuity of the functions involved in (5) and relation $K = \overline{\operatorname{Int} K}$ imply that (5) should hold everywhere on K. Set

$$f^* = f - \frac{1}{k+2} \sum_{i=0}^{k+1} q_i$$

and denote $Z^* = Z(q_1^*, \ldots, q_{k+1}^*)$. Evidently, 0 is a best approximant of f^*. Moreover, by (5) we have $Z(f^*) \subset Z^*$. Let ψ be equal to sign f^* and 0 on $K \backslash Z^*$ and Z^*, respectively. Since $Z(f^*) \subset Z^*$ it follows easily that ψ is a sign function with $\operatorname{supp} \psi = K \backslash Z^*$. It is known (see, e.g., [**17**, p. 46]) that in order for 0 to be a best approximant of f^* it is necessary and sufficient that for every $q \in U_n$

$$\left| \int_{K \backslash Z(f^*)} \omega q \operatorname{sign} f^* \right| \leq \int_{Z(f^*)} \omega |q|. \tag{6}$$

This and $Z(f^*) \subset Z^*$ imply that $|\int_{K \backslash Z^*} \omega \psi q| \leq \int_{Z^*} \omega |q|$, $q \in U_n$. But this last statement means that for the sign function ψ with $\operatorname{supp} \psi = K \backslash Z(q_1^*, \ldots, q_{k+1}^*)$ there does not exist $q \in U_n \backslash \{0\}$ such that $\psi q \geq 0$ on K and $q = 0$ a.e. on Z^*. Hence U_n is not an A^k-space.

Necessity. Now let U_n have Chebyshev rank at most k in $C_\omega(K)$ for every $\omega \in W$. Consider any linearly independent elements $q_0, q_1, \ldots, q_k \in U_n$ and sign function ψ with $\operatorname{supp} \psi = K \backslash Z(q_0, \ldots, q_k)$. First we shall prove that for any $\omega \in W$ there exists $q \in U_n$ for which

$$\int_{K \backslash Z(q_0, \ldots, q_k)} \omega \psi q > \int_{Z(q_0, \ldots, q_k)} \omega |q|. \tag{7}$$

Assume that to the contrary for some $\tilde{\omega} \in W$

$$\left| \int_{K \backslash Z(q_0, \ldots, q_k)} \tilde{\omega} \psi q \right| \leq \int_{Z(q_0, \ldots, q_k)} \tilde{\omega} |q|, \qquad q \in U_n. \tag{8}$$

Set $q^* = \psi \max_{0 \leq i \leq k} |q_i|$. Then $q^* \in C(K)$, $Z(q^*) = Z(q_0, \ldots, q_k)$ and $\operatorname{sign} q^* = \psi$ on K. Again using criteria (6) and (8) we can conclude that 0 is a best approximant of $q^* \in C_{\tilde{\omega}}(K)$. Furthermore, for any $0 < \varepsilon < 1$ and $x \in K$ we have

$$\operatorname{sign}(q^* - \varepsilon q_i)(x) = \operatorname{sign} q^*(x), \qquad 0 \leq i \leq k.$$

Thus εq_i $(0 \leq i \leq k)$ are also best approximants of $q^* \in C_{\tilde{\omega}}(K)$, i.e., $\dim Y(q^*) \geq k+1$ (with respect to weight $\tilde{\omega}$), which contradicts our assumption on the Chebyshev rank of U_n in $C_{\tilde{\omega}}(K)$. Hence for any $\omega \in W$ there should exist $q \in U_n$ satisfying (7).

Set $\tilde{U} = \{q \in U_n : q = 0 \text{ a.e. at } Z(q_0, \ldots, q_k)\}$, $k+1 \leq \dim \tilde{U} \leq n$. Assume that the space $U^* = \psi \tilde{U} = \{\psi \tilde{q} : \tilde{q} \in \tilde{U}\}$ does not contain a nonnegative element (a.e. at K). Then by Lemma 1 there exists $\tilde{\omega} \in W$ such that for every $q^* = \psi \tilde{q} \in U^*$ $(\tilde{q} \in \tilde{U})$ we have

$$0 = \int_K \tilde{\omega} q^* = \int_{K \backslash Z(q_0, \ldots, q_k)} \tilde{\omega} \psi \tilde{q}, \qquad \tilde{q} \in \tilde{U}. \tag{9}$$

Let U_n be the direct sum of $\tilde{U}$ and a subspace $U_1 \subset U_n$, $U_n = \tilde{U} + U_1$. Since elements of U_1 cannot vanish a.e. on $Z(q_0, \ldots, q_k)$ there exists a constant $\eta > 0$ such that for each $q_1 \in U_1$

$$\max_{x\in K} |q_1(x)| \leq \eta \int_{Z(q_0,\ldots,q_k)} |q_1|. \tag{10}$$

Evidently, we may assume that

$$\tilde{\omega}(x) = \eta \operatorname{mes}(K) \sup\{\tilde{\omega}(x): x \in K\backslash Z(q_0, \ldots, q_k)\}$$

for $x \in Z(q_0, \ldots, q_k)$; this will not violate relation (9). Then for any $q = \tilde{q}+q_1 \in U_n$, where $\tilde{q} \in \tilde{U}$, $q_1 \in U_1$, we obtain by (9) and (10)

$$\int_{K\backslash Z(q_0,\ldots,q_k)} \tilde{\omega}\psi q = \int_{K\backslash Z(q_0,\ldots,q_k)} \tilde{\omega}\psi q_1 \leq \int_{Z(q_0,\ldots,q_k)} \tilde{\omega}|q_1| = \int_{Z(q_0,\ldots,q_k)} \tilde{\omega}|q|.$$

But this contradicts (7). Hence U^* should contain a nonnegative element, i.e., $\psi\tilde{q} \geq 0$ a.e. on K for some $\tilde{q} \in \tilde{U}\backslash\{0\}$. Moreover, by construction of $\tilde{U} \subset U_n$ we also have that $\tilde{q} = 0$ a.e. on $Z(q_0, \ldots, q_k)$, i.e. the A^k-property of U_n is verified.

4. Haar spaces at disjoint intervals. The following property of Haar spaces plays a crucial role in the proof of the Jackson-Krein theorem: if $U_n \subset C[a, b]$ is a Haar space, then for any $x_1, \ldots, x_k \in (a, b)$, where $0 \leq k \leq n-1$, there exists a $q \in U_n$ vanishing in (a, b) only at the points x_i, $1 \leq i \leq k$, and changing its sign at each of them. Thus given any sign function with at most $n-1$ zeros in (a, b) and changing its sign at each of them we can find an element in U_n having the same sign in (a, b). In particular, this yields existence of positive elements in U_n. One may ask whether these properties are preserved in case of a Haar space at disjoint intervals. Consider for instance the one-dimensional space $U_1 = \{\alpha \operatorname{sign} x: \alpha \in \mathbf{R}\}$, where $x \in [-2, -1] \cup [1, 2]$. U_1 satisfies the Haar property on $[-2, -1] \cup [1, 2]$ but does not contain a nonnegative element on this set. Nevertheless, it turns out that if *the number of disjoint intervals does not exceed the dimension of* U_n (that is n) then there exists a natural extension of the above properties to the case of disjoint intervals.

Set $K_m = \bigcup_{j=1}^m [\alpha_j, \beta_j]$, where $m \in \mathbf{N}$ and $\alpha_1 < \beta_1 < \alpha_2 < \cdots < \alpha_m < \beta_m$; $\operatorname{Int} K_m = \bigcup_{j=1}^m (\alpha_j, \beta_j)$.

THEOREM 4. *Let $U_n \subset C(K_m)$ satisfy the Haar property on K_m and assume that $1 \leq m \leq n$. Then for any sign function ψ having at most $n-m$ zeros in* $\operatorname{Int} K_m$ *and changing its sign at each of them there exists $q \in U_n$ such that* $\operatorname{sign} q = \psi$ *in* $\operatorname{Int} K_m$.

This theorem implies the following result of Mandler [**21**].

COROLLARY 1. *If $U_n \subset C(K_m)$ is a Haar space and $1 \leq m \leq n$, then U_n contains an element which is positive on* $\operatorname{Int} K_m$.

First of all let us make one observation. If U_n is a Haar space on K_m, then the number of zeros of each $q \in U_n\backslash\{0\}$ on K_m counting nonnodal zeros twice does not exceed $n-1$. (Recall that $x \in \operatorname{Int} K_m$ is a nonnodal zero of q if q does not change its sign at x.) This statement can be verified in the same manner as in the case of one interval (see [**10**, p. 54]).

Before proving Theorem 4 we introduce some additional notations and verify two simple lemmas. Let $\varphi_1, \dots, \varphi_n$ be a basis in U_n. For any $X_n = \{x_1, \dots, x_n\} \subset K_m$ set

$$D(X_n) = D(x_1, \dots, x_n) = \det\{\varphi_i(x_j)\}_{i=1,\dots,n;\ j=1,\dots,n}.$$

Obviously, U_n is a Haar space on K_m iff for any distinct points $x_1, \dots, x_n \in K_m$ we have $D(x_1, \dots, x_n) \neq 0$. But in contrast with the case of one interval in our situation the determinants $D(x_1, \dots, x_n)$ may have opposite signs. For any $X_n = \{x_1, \dots, x_n\} \subset K_m$ denote by $r_j(X_n)$ the number of points in X_n belonging to $[\alpha_j, \beta_j]$, $1 \leq j \leq m$, and set $R(X_n) = \{r_j(X_n)\}_{j=1}^m \in \mathbf{Z}_+^m$. The next two lemmas easily follow from definitions and therefore we leave the proof to the reader.

LEMMA 2. *Let $X_n = \{x_1 < \cdots < x_n\}$ and $X_n^* = \{x_1^* < \cdots < x_n^*\}$ be two ordered systems of n points in K_m. If $R(X_n) = R(X_n^*)$, then* $\operatorname{sign} D(X_n) = \operatorname{sign} D(X_n^*)$.

LEMMA 3. *Let $x_1 < \cdots < x_{n-1}$ be in* $\operatorname{Int} K_m$. *Then $q(x) = D(x_1, \dots, x_{n-1}, x) \in U_n$ vanishes only at the points x_i, $1 \leq i \leq n-1$, and changes its sign at each of them.*

REMARK. Note that if x_i and x_{i+1} belong to different intervals in K_m, then q may have a sign change in between x_i and x_{i+1}. This is the principal difference between the cases of one and several intervals.

PROOF OF THEOREM 4. Assume that $x_1, \dots, x_k \in \operatorname{Int} K_m$ $(0 \leq k \leq n-m)$ are the points where ψ has its sign changes. We shall now extend this point set to a set of points of length $n-1$ in the following way: we add one point to each interval (α_j, β_j), $2 \leq j \leq m-1$, and $n-k-m+1$ points to (α_m, β_m). The location of the new points in corresponding intervals will be specified below, at this stage we only assume that all of them differ from x_i, $1 \leq i \leq k$. Thus we get a system of $n-1$ nodes $x_1^* < \cdots < x_{n-1}^*$ in $\operatorname{Int} K_m$. Since $k \leq n-m$ at least one point was added to (α_m, β_m) as well. Set

$$\gamma_j = \operatorname{sign} D(x_1^*, \dots, x_{n-1}^*, \alpha_j), \qquad 1 \leq j \leq m.$$

Observe that the γ_j's are independent of the way how we locate the new points in corresponding intervals. This easily follows from Lemma 2. Hence we have absolute freedom in moving the new points inside the corresponding intervals without violating the γ_j's. Set $\psi_j = \psi(\alpha_j)$, $1 \leq j \leq m$, and let $\varepsilon > 0$ be smaller than the minimal distance between any of the x_k's and the boundary points of K_m. Now we shall specify the location of new points in (α_j, β_j), $2 \leq j \leq m-1$. We add to (α_j, β_j) the point $\beta_j - \varepsilon$ if $\psi_j = \gamma_1 \psi_1 \gamma_j$ and the point $\alpha_j + \varepsilon$ if $\psi_j = -\gamma_1 \psi_1 \gamma_j$ $(2 \leq j \leq m-1)$. We still have to determine $n-k-m+1$ new points in (α_m, β_m). If $\psi_m = -\gamma_1 \psi_1 \gamma_m$ we add the point $\alpha_m + \varepsilon$ plus points $\xi_i, \xi_i + \varepsilon$ $(1 \leq i \leq s;\ s = [(n-k-m)/2])$, where the ξ_i's are arbitrary fixed nodes such that $\max\{x_k, \alpha_m + \varepsilon\} < \xi_1 < \xi_1 + \varepsilon < \xi_2 < \cdots < \xi_s < \xi_s + \varepsilon < \beta_m - \varepsilon$. In the case when $n-k-m$ is odd we also add the point $\beta_m - \varepsilon$. Analogously, if $\psi_m = \gamma_1 \psi_1 \gamma_m$ we add points $\xi_i, \xi_i + \varepsilon$ $(1 \leq i \leq s_1;\ s_1 = [(n-k-m+1)/2])$, where the ξ_i's are chosen as above, plus $\beta_m - \varepsilon$ if $n-k-m+1$ is odd. Now the system of nodes $x_1^*, \dots, x_{n-1}^*$ is completely specified. Set $q_\varepsilon(x) = \gamma_1 \psi_1 c_\varepsilon D(x_1^*, \dots, x_{n-1}^*, x)$, where $c_\varepsilon > 0$ is chosen in such a way that $\max\{|q_\varepsilon(x)| : x \in K_m\} = 1$. Evidently, $\operatorname{sign} q_\varepsilon(\alpha_j) = \gamma_1 \psi_1 \gamma_j$, $1 \leq j \leq m$. Therefore Lemma 3 yields that $\operatorname{sign} q_\varepsilon(x) = \psi(x)$

for any $x \in (\alpha_j + \varepsilon, \beta_j - \varepsilon)$ and $1 \le j \le m-1$. Finally, consider the interval (α_m, β_m). Again using Lemma 3 and our choice of additional points in (α_m, β_m) we easily obtain that $\operatorname{sign} q_\varepsilon(x) = \psi(x)$ for all $x \in (\alpha_m + \varepsilon, \beta_m - \varepsilon)$ except if $x \in [\xi_i, \xi_i + \varepsilon]$ $(1 \le i \le s$ or s_1, respectively). Since $\max\{|q_\varepsilon(x)|: x \in K_m\} = 1$ we may assume without loss of generality that the q_ε tend uniformly to $q_0 \in U_n \backslash \{0\}$ as $\varepsilon \to +0$.

Taking into account that the total number of zeros of q_0 on K_m counting non-nodal zeros twice does not exceed $n-1$ we easily obtain that inside K_m q_0 vanishes only at $x_1, \ldots, x_k$ and ξ_i $(1 \le i \le s$ or $s_1)$. Moreover, the sign property of $q_\varepsilon(x)$ implies that $\operatorname{sign} q_0(x) = \psi(x)$ for $x \in \operatorname{Int} K_m$, $x \ne \xi_i$ $(1 \le i \le s$ or $s_1)$. Since we have certain freedom in choosing the ξ_i's we can similarly construct a $q^* \in U_n \backslash \{0\}$ such that $\operatorname{sign} q^* = \psi$ for $x \in \operatorname{Int} K_m$, $x \ne \xi_i^*$, $q^*(\xi_i^*) = 0$ $(1 \le i \le s$ or $s_1)$ where all the ξ_i^*'s are distinct from the ξ_i's. Then $\operatorname{sign}(q_0 + q^*) = \psi$ in $\operatorname{Int} K_m$. The theorem is proved.

Corollary 1 follows by setting $\psi \equiv 1$ in Theorem 4.

Now, using Theorem 4 we can easily verify the next statement.

COROLLARY 2. *Let $U_n \subset C(K_m)$ satisfy the Haar property on* Int K_m, *where* $1 \le m \le n$. *Then U_n is an A^{m-1}-space.*

PROOF. Consider any m linearly independent elements $q_0, q_1, \ldots, q_{m-1} \in U_n$ and a sign function ψ with $\operatorname{supp} \psi = K_m \backslash Z(q_0, \ldots, q_{m-1})$. Evidently, $Z(q_0, \ldots, q_{m-1})$ consists of at most $n-m$ points, that is, without loss of generality we may assume that ψ satisfies the condition of Theorem 4. Further, set $K_m^\varepsilon = \bigcup_{j=1}^m [\alpha_j + \varepsilon, \beta_j - \varepsilon]$, where $\varepsilon > 0$ is sufficiently small. Applying Theorem 4 at K_m^ε and using the fact that U_n satisfies the Haar property on K_m^ε we can find $q_\varepsilon \in U_n$ with supremum norm equal to 1 such that $\operatorname{sign} q_\varepsilon = \psi$ in $\operatorname{Int} K_m^\varepsilon$. Letting $\varepsilon \to +0$ (and passing to a subsequence if necessary) we get a $q_0 \in U_n \backslash \{0\}$ satisfying $\psi q_0 \ge 0$ on K_m. Thus we have verified the A^{m-1}-property of U_n.

The above corollary and Theorem 2 immediately imply Theorem 3 stated in §1.

Now we shall investigate the following question: Can a Haar space $U_n \subset C(K_m)$ have Chebyshev rank less than $m-1$ in $C_\omega(K_m)$? It will be shown that the answer to this question is negative, that is, Theorem 3 gives a sharp estimate of Chebyshev rank.

First of all we prove a lemma based on an idea of Havinson [**4**].

LEMMA 4. *Let U_n be an arbitrary n-dimensional subspace in $L_1(S)$, where $S \subset \mathbf{R}^s$ is a measurable bounded set. Consider $n+1$ pairwise disjoint measurable subsets $A_1, \ldots, A_{n+1}$ of S and assume that*

$$\mu\{x \in A_j: q(x) \ne 0\} > 0 \tag{11}$$

for any $q \in U_n \backslash \{0\}$ and $1 \le j \le n+1$. Then there exists $\omega \in W$ and $\gamma_j \in \mathbf{R}$, $|\gamma_j| = 1$ $(1 \le j \le n+1)$ such that for every $q \in U_n$

$$\sum_{j=1}^{n+1} \gamma_j \int_{A_j} \omega q = 0. \tag{12}$$

PROOF. Let $U_n = \operatorname{span}\{\varphi_1, \dots, \varphi_n\}$ and set

$$M_{n,n+1} = \left\{ \left(\int_{A_j} \omega\varphi_i \right)_{i=1,\dots,n;\ j=1,\dots,n+1} : \omega \in W \right\}.$$

Since $M_{n,n+1}$ consists of $n \times (n+1)$ matrices we have $M_{n,n+1} \subset \mathbf{R}^{n(n+1)}$. Furthermore, $M_{n,n+1}$ is a convex subset of $\mathbf{R}^{n(n+1)}$ and assumption (11) implies that the interior of $M_{n,n+1}$ in $\mathbf{R}^{n(n+1)}$ is nonempty. It was shown in [**4**] that the set of $n \times (n+1)$ matrices having all their $n \times n$ determinants nonvanishing is dense in $\mathbf{R}^{n(n+1)}$. Therefore, for some $\omega^* \in W$ the matrix $(\int_{A_j} \omega^*\varphi_i)_{i=1,\dots,n;\ j=1,\dots,n+1}$ has all of its $n \times n$ determinants nonzero, yielding that the system of equations

$$\sum_{j=1}^{n+1} a_j \int_{A_j} \omega^* \varphi_i = 0, \qquad 1 \le i \le n,$$

has nonvanishing solutions $a_j \in \mathbf{R}\backslash\{0\}$, $1 \le j \le n+1$. Setting $\omega(x) = |a_j|\omega^*(x)$ for $x \in A_j$ and $\gamma_j = \operatorname{sign} a_j$ $(1 \le j \le n+1)$ we obtain (12).

In what follows we shall say that the subspace $U_n \subset C(K)$ satisfies the Z^*-property if no nontrivial element of U_n vanishes on a nonempty open subset of K.

THEOREM 5. *Let $U_n \subset C(K)$ be an A^k-space satisfying the Z^*-property $(n \ge 1,\ 0 \le k \le n-1)$. Then for any $k+1$ linearly independent elements $q_0, q_1, \dots, q_k \in U_n$ the set $K\backslash Z(q_0, \dots, q_k)$ is at most n-disconnected.*

PROOF. Assume that to the contrary $K\backslash Z(q_0, \dots, q_k) = A_1 \cup \dots \cup A_{n+1}$, where the A_j's are open (in $K\backslash Z(q_0, \dots, q_k)$), nonempty, pairwise disjoint sets. Using Lemma 4 we can find $\omega \in W$ and $\gamma_j \in \mathbf{R}$, $|\gamma_j| = 1$ $(1 \le j \le n+1)$ for which (12) holds for every $q \in U_n$. Then setting ψ equal to γ_j on A_j $(1 \le j \le n+1)$ and to zero on $Z(q_0, \dots, q_k)$ we obtain a sign function with $\operatorname{supp}\psi = K\backslash Z(q_0, \dots, q_k)$. By the A^k-property of U_n there should exist $\tilde{q} \in U_n\backslash\{0\}$ such that $\psi\tilde{q} \ge 0$ on K and $\tilde{q} = 0$ a.e. on $Z(q_0, \dots, q_k)$, which implies that (12) does not hold for $\tilde{q}$, a contradiction.

Note that in [**13**] we proved a more general version of Theorem 5 in the case when $k = 0$.

THEOREM 6. *Let $K \subset \mathbf{R}$ and assume that* $\operatorname{Int} K$ *is k-disconnected. If $U_n \subset C(K)$ satisfies the Z^*-property $(2 \le k \le n+1)$, then U_n is not an A^{k-2}-space.*

PROOF. Let $\operatorname{Int} K = \bigcup_{j=1}^k A_j$, where the A_j's are pairwise disjoint open sets, and choose any $x_1, \dots, x_{n-k+1} \in \operatorname{Int} K$. Obviously, there exist $k-1$ linearly independent elements $q_0, \dots, q_{k-2} \in U_n$ such that $Z(q_0, \dots, q_{k-2}) \supset \{x_1, \dots, x_{n-k+1}\}$. Assume that U_n is an A^{k-2}-space. Then by Theorem 5, $K\backslash Z(q_0, \dots, q_{k-2})$ should be at most n-disconnected. But $\operatorname{Int} K\backslash Z(q_0, \dots, q_{k-2}) \subset \operatorname{Int} K\backslash\{x_1, \dots, x_{n-k+1}\}$, hence the Z^*-property of U_n implies that $\operatorname{Int} K\backslash Z(q_0, \dots, q_{k-2})$ is at least $n+1$-disconnected, contradicting Theorem 5.

COROLLARY 3. *Let $K \subset \mathbf{R}$ and assume that* $\operatorname{Int} K$ *is m-disconnected. If $U_n \subset C(K)$ is a Haar space on* $\operatorname{Int} K$*, then U_n is not an A^{m-2}-space $(2 \le m \le n+1)$.*

In the special case $m = 2$, Corollary 3 was verified by Sommer [**18**].

Combining the above statement with Corollary 2 we can conclude that if $U_n \subset C(K_m)$ satisfies the Haar property on $\operatorname{Int} K_m$ then U_n is an A^{m-1}-space but not an A^{m-2}-space $(2 \le m \le n,\ K_m = \bigcup_{j=1}^{m}[\alpha_j, \beta_j])$. Thus for some $\tilde{\omega} \in W$ the Chebyshev rank of U_n in $C_{\tilde{\omega}}(K_m)$ is exactly $m-1$, showing the sharpness of Theorem 3.

Theorem 5 also implies

COROLLARY 4. *Let $U_n \subset C(K_m)$ be an A^{m-1}-space satisfying the Z^*-property $(1 \le m \le n)$. Then any m linearly independent elements of U_n have at most $n-m$ common zeros inside K_m.*

PROOF. If for some linearly independent elements $q_0, \dots, q_{m-1} \in U_n$ the set $Z(q_0, \dots, q_{m-1})$ contains more than $n-m$ inner points of $\operatorname{Int} K_m$ then it follows by the Z^*-property that $K_m \setminus Z(q_0, \dots, q_{m-1})$ is at least $n+1$-disconnected, contradicting Theorem 5.

Recall that by Rubinstein's theorem if any m linearly independent elements of U_n have at most $n-m$ common zeros on K then the Chebyshev rank of U_n in $C(K)$ is at most $m-1$. Thus Corollary 4 implies that if U_n satisfies the Z^*-property and has Chebyshev rank at most $m-1$ in $C_\omega(K_m)$ for every $\omega \in W$ then its Chebyshev rank in $C(\tilde{K})$, where $\tilde{K}$ is an arbitrary compact subset of $\operatorname{Int} K_m$, also does not exceed $m-1$. In the special case $m=1$ this implies the known result of Havinson [**4**]: if U_n is a Chebyshev subspace of $C_\omega[a,b]$ for each $\omega \in W$ and satisfies the Z^*-property, then U_n is a Haar space on (a,b).

5. Applications. In this section we shall apply our results and present various examples of A^k-spaces.

PROPOSITION 2. *Let $U_n \subset C(K)$ contain an $(n-k)$-dimensional A-space, $0 \le k \le n-1$. Then U_n is an A^k-space.*

PROOF. Consider arbitrary, linearly independent elements $q_0, \dots, q_k \in U_n$, and let $\tilde{U}_{n-k}$ be an $(n-k)$-dimensional A-subspace of U_n. Then, evidently, $\tilde{U}_{n-k} \cap \operatorname{span}\{q_0, \dots, q_k\} \ne \emptyset$, i.e., there exists $\tilde{q} \in \operatorname{span}\{q_0, \dots, q_k\} \setminus \{0\}$ such that $\tilde{q} \in \tilde{U}_{n-k}$. Furthermore, we have $Z(q_0, \dots, q_k) \subset Z(\tilde{q})$. Let ψ be an arbitrary sign function with $\operatorname{supp} \psi = K \setminus Z(q_0, \dots, q_k)$. Consider also the sign function $\tilde{\psi}$ equal to ψ on $K \setminus Z(\tilde{q})$ and to 0 on $Z(\tilde{q})$. Evidently, $\operatorname{supp} \tilde{\psi} = K \setminus Z(\tilde{q})$. Since $\tilde{q} \in \tilde{U}_{n-k}$ and $\tilde{U}_{n-k}$ is an A-space there exists $q \in \tilde{U}_{n-k} \setminus \{0\}$ such that $\tilde{\psi} q \ge 0$ on K and $q = 0$ a.e. at $Z(\tilde{q})$. This implies that $\psi q \ge 0$ on K and $q = 0$ a.e. on $Z(q_0, \dots, q_k)$, verifying the A^k-property of U_n.

COROLLARY 5. *If $U_n \subset C[a,b]$ contains an $(n-k)$-dimensional Haar subspace, then U_n is an A^k-space $(0 \le k \le n-1)$.*

Denote by $P_{n+1} = \operatorname{span}\{1, x, \dots, x^n\}$ the space of algebraic polynomials of degree at most n. If we delete a basis function x^j $(1 \le j \le n-1)$ then the resulting space given by $\operatorname{span}\{1, \dots, x^{j-1}, x^{j+1}, \dots, x^n\}$ does not satisfy the Haar property on $[-1,1]$, thus it is not an A-space as well. Nevertheless, it can be shown that this space of lacunary polynomials with one "gap" is an A^1-space on $[-1,1]$. We shall prove below that in general a lacunary polynomial space with k "gaps" satisfies the A^k-property. Let $1 \le r_1 < \cdots < r_k \le n-1$ be arbitrary integers $(1 \le k \le n-1)$.

Set

$$P_{n+1-k} = \text{span}\{x^j : 0 \le j \le n;\ j \ne r_i,\ 1 \le i \le k\}. \tag{13}$$

THEOREM 7. *Let P_{n+1-k} be a polynomial space of the form (13) with k "gaps", $1 \le k \le [n/2]$. Then P_{n+1-k} is an A^k-space on $[-1,1]$ and, in general, it does not satisfy the A^{k-1}-property on $[-1,1]$.*

PROOF. It is known (see, e.g. [**22**]) that $\text{span}\{x^{m_0}, x^{m_1}, \dots, x^{m_s}\}$, where $0 = m_0 < m_1 < \cdots < m_s$ are arbitrary integers, is a Haar space on $[-1,1]$ if $m_{i+1} - m_i$ is odd for every $0 \le i \le s-1$. Therefore, deleting from P_{n+1-k} at most k basis functions we can obtain a Haar space on $[-1,1]$ (see [11] for details). Thus P_{n+1-k} contains a Haar subspace of dim $\ge n+1-2k$, i.e., by Corollary 5 it satisfies the A^k-property.

In order to show that P_{n+1-k} cannot, in general, be an A^{k-1}-space on $[-1,1]$ we shall consider a special choice of integers $r_1, \dots, r_k$ in (13). Set $r = n+1-2k$ $(1 \le r \le n-1)$ and let

$$\tilde{P}_{n+1-k} = \text{span}\{1, \dots, x^{r-1}, x^{r+1}, x^{r+3}, \dots, x^{r-1+2k} = x^n\},$$

i.e., $r_i = r + 2(i-1)$, $1 \le i \le k$. We state that $\tilde{P}_{n+1-k}$ is not an A^{k-1}-space on $[-1,1]$. Let $-1 < \xi_1 < \cdots < \xi_r < 1$ be the nodes of the Chebyshev polynomial of the second kind of degree r. Assume that r is even (the case of r odd can be treated analogously). It is well known that

$$\int_{-1}^{1} \psi(x) x^j \, dx = 0, \qquad 0 \le j \le r-1, \tag{14}$$

where $\psi(x) = (-1)^i$, $x \in (\xi_i, \xi_{i+1})$ $(0 \le i \le r;\ \xi_0 = -1,\ \xi_{r+1} = 1)$. Moreover, since r is even it follows that ψ is an even sign function, while the powers x^{r-1+2j} are odd. Therefore,

$$\int_{-1}^{1} \psi(x) x^{r-1+2j} \, dx = 0, \qquad 1 \le j \le k.$$

This together with (14) implies that for any $q \in \tilde{P}_{n+1-k}$

$$\int_{-1}^{1} \psi(x) q(x) \, dx = 0. \tag{15}$$

Therefore, setting $\psi(\xi_i) = 0$, $1 \le i \le r$, we obtain a sign function with $\text{supp}\,\psi = [-1,1] \backslash \{\xi_1, \dots, \xi_r\}$ while by (15) there does not exist $q \in \tilde{P}_{n+1-k} \backslash \{0\}$ such that $q\psi \ge 0$ on $[-1,1]$. On the other hand $\dim \tilde{P}_{n+1-k} = n+1-k = r+k$, yielding that there exist k linearly independent elements of $\tilde{P}_{n+1-k}$ vanishing at ξ_i, $1 \le i \le r$. Thus $\tilde{P}_{n+1-k}$ is not an A^{k-1}-space on $[-1,1]$.

Now we shall consider A^k-spaces of a different type, not necessarily containing A-subspaces (or Haar subspaces) of smaller dimension.

PROPOSITION 3. *Let $U_n \subset C[a,b]$ be a Haar space on (a,b) and assume that the function $\varphi \in C[a,b]$ has k distinct zeros in (a,b). Then $\tilde{U}_n = \varphi U_n = \{\varphi q : q \in U_n\}$ is an A^k-space and not an A^{k-1}-space $(1 \le k \le n-1)$.*

PROOF. Consider $\varphi q_0, \dots, \varphi q_k \in \tilde{U}_n$, where $q_0, \dots, q_k \in U_n$ are linearly independent. Obviously, $Z(\varphi q_0, \dots, \varphi q_k)$ contains at most $(n-k-1)+k = n-1$

inner points of (a,b). Let ψ be an arbitrary sign function with $\operatorname{supp}\psi = [a,b]\backslash Z(\varphi q_0,\dots,\varphi q_k)$, where $Z(\varphi q_0,\dots,\varphi q_k) = \{x_1,\dots,x_s\}$, $k \le s \le n-1$. We may assume that $x_1,\dots,x_k$ are the zeros of φ and $x_1,\dots,x_r$, $0 \le r \le k$, are the points of sign change of φ. Now choose $q \in U_n$ in such a way that it changes its sign precisely at the following points: those $x_1,\dots,x_r$ where ψ does not change its sign and those $x_{r+1},\dots,x_s$ where ψ changes its sign. This choice of q is possible because we have prescribed at most $n-1$ sign changes. Now, evidently, $\varphi q \in \tilde{U}_n$ changes its sign only at the points of sign change of ψ. Therefore, $\tilde{U}_n$ is an A^k-space.

In order to prove that $\tilde{U}_n$ is not an A^{k-1}-space let us choose arbitrary points $x_1^*,\dots,x_{n-k}^* \in (a,b)$ different from x_i, $1 \le i \le k$. There exist linearly independent elements $q_1,\dots,q_k \in U_n$ such that $Z(q_1,\dots,q_k) = \{x_1^*,\dots,x_{n-k}^*\}$. Thus $\varphi q_1,\dots,\varphi q_k \in \tilde{U}_n$ will have n distinct common zeros $x_1^*,\dots,x_{n-k}^*, x_1,\dots,x_k \in (a,b)$. Let ψ be a sign function with $\operatorname{supp}\psi = [a,b]\backslash Z(\varphi q_1,\dots,\varphi q_k)$ changing its sign at $x_1^*,\dots,x_{n-k}^*$ and those x_i's $(1 \le i \le k)$ where φ does not change its sign. Assume that for some $\tilde{q} \in \tilde{U}_n\backslash\{0\}$ $(\tilde{q} = \varphi q,\ q \in U_n)$ the inequality $\tilde{q}\psi \ge 0$ holds. Then $\psi\varphi q \ge 0$ and since $\psi\varphi$ has n sign changes it follows that q should also change its sign n times, contradicting the Haar property of U_n. Thus $\tilde{U}_n$ is not an A^{k-1}-space.

Let us give an example of an application of Proposition 3. Consider arbitrary distinct points $x_1,\dots,x_k \in (a,b)$ and denote by $P_{n+1}^{(k)}$ the set of algebraic polynomials of degree at most $n+k$ vanishing at $x_1,\dots,x_k$, i.e., $P_{n+1}^{(k)} = \{p \in P_{n+k+1} : p(x_1) = \cdots = p(x_k) = 0\} = \tilde{\varphi}P_{n+1}$, where $\tilde{\varphi}(x) = (x-x_1)\cdots(x-x_k)$. Then we can easily derive

COROLLARY 6. *For any $1 \le k \le n-1$ the set $P_{n+1}^{(k)}$ is an A^k-space on $[a,b]$ and not an A^{k-1}-space, i.e., the Chebyshev rank of $P_{n+1}^{(k)}$ in $C_\omega[a,b]$ is at most k for every $\omega \in W$ and is precisely k for some $\tilde{\omega} \in W$.*

Proposition 3 unveils a surprising fact. Namely, the spaces $\tilde{U}_n = \varphi U_n$ considered in Proposition 3 do not satisfy the condition of Rubinstein's theorem with any integer $1 \le k \le n$. Therefore, their Chebyshev rank with respect to the C-norm at $[a,b]$ is equal to n. Thus at $[a,b]$ A^k-spaces with the Z^*-property do not necessarily have Chebyshev rank k with respect to the C-norm if $k \ge 1$. This contrasts the case $k=0$ since A-spaces with the Z^*-property are necessarily Haar spaces.

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